NOTE ON MR. GEORGE PEIRCE'S APPROXIMATE CONSTRUCTION FOR π.

BY M. EMILE LEMOINE.

(Read before the American Mathematical Society, August 20, 1901.)

THERE appeared in the BULLETIN for July, 1901, a construction by Mr. George Peirce for obtaining the approximate length of π in a circle of radius 1. There are numerous constructions of this kind, and it may be of interest to indicate the method which permits a comparison of these constructions with one another as to their graphical simplicity. Their relative theoretic exactness is determined by calculating the true value of the length which in each case approximately represents π.

As examples of these comparisons, I take the construction of Mr. Peirce and three others, and employ the geometrographic method (see Mathematical papers of the Chicago congress, 1893, p. 143, or in more complete form, La géométrografie, Paris, Naud, 1901) which is applicable with rigor and facility. I will designate by :

A. The construction of Mr. Peirce.

B A very old construction, attributed to Heinrich Kühn, in the *Novi Commentarii Acad. Petropol.*, Vol. III (1753).

C. A construction given by myself for $\pi/2$.

D. A construction due to Professor Pleskot of the Czech Realschule of Prague (*Journal de mathématiques élémentaires* de M. de Longchamps, 1895, p. 125); this also gives $\pi/2$.

The geometrographic notation is so simple that I indicate it at once, so that any geometer not acquainted with it may have no difficulty in comprehending this note.

1. Placing *one* point of the compass on a given point is designated as "operation C_1" or op (C_1); hence, speculatively, including a given length between the points is op. $(2C_1)$.

2. Placing a point of the compass on an *undetermined* point of a straight line is op. (C_2).

3. Drawing a circle is op. (C_3).

4. Making the edge of the ruler pass through *one* point is op. (R_1); hence, speculatively, making it pass through two points is op. $(2R_1)$.

5. Drawing a straight line is op. (R_2).

This is all for the *canonical* geometrography, *i. e.*, where the only instruments used are ruler and compass.

If the square is also admitted, the same notation is retained as for the operations with the ruler, but the R is accented : op. (R_1'), op. $(2R_1')$. I also accent any operation with the ruler that serves *directly* for a construction requiring the square. This is done merely for the purpose of making the symbols themselves indicate the extent to which the square is used.

Placing the edge of the square (or ruler) in coincidence with a given straight line, an operation which never occurs in canonical geometrography, is regarded as passing this edge through two points and denoted by op. $(2R_1')$. Finally, sliding one side of the square along the ruler until the other side passes through a given point is op. (E). This is the only symbol peculiar to the square.

Any canonical construction is thus represented by a symbol of the form op. $(l_1R_1 + l_2R_2 + m_1C_1 + m_2C_2 + m_3C_3)$. The sum $l_1 + l_2 + m_1 + m_2 + m_3$ is called the coefficient of simplicity or simply the *simplicity*. The sum $l_1 + m_1 + m_2$ of the coefficients of the *preparatory* operations is called the coefficient of exactness or simply the *exactness* ; l_2 and m_3 are the number of straight lines and circles drawn.

When the square is admitted, the symbol of a construction will be similarly

$$\text{op. } (l_1R_1 + l_1'R_1' + l_2R_2 + kE + m_1C_1 + m_2C_2 + m_3C_3);$$

$l_1 + l_1' + l_2 + k + m_1 + m_2 + m_3$ is the simplicity, $l_1 + l_1' + k + m_1 + m_2$ the exactness ; l_2 and m_3 are the number of straight lines and circles drawn. For brevity, a circle with center A and radius R or MN may be denoted by $A(R)$ or $A(MN)$.

NOTE.—Geometrography is essentially speculative ; it assumes, for instance, like geometry, that the lines and circles are completely drawn, that the paper is a plane of infinite extent, that the drawing instruments are as large or as small as may be necessary, etc. It therefore guides the draughtsman as rational mechanics guides the engineer, but *far* more exactly.

EXAMINATION OF THE CONSTRUCTIONS.

A. (*Figure* 1.)—At the outset, the figure in the plane consists only of a circle of radius 1 with marked center O.

1°. *Canonical.*—Draw any diameter $AC(R_1 + R_2)$. By means of the intersection of the two circles $A(\rho)$, $C(\rho)(2C_1 + 2C_3)$, ρ being any sufficient length, draw the diameter $BD(2R_1 + R_2)$ perpendicular to AC. Place one point of the compass on D, the other on O, and draw $D(DO)(2C_1 + C_3)$. Draw $AD(2R_1 + R_2)$ which cuts $D(DO)$ in E, between A and D. Draw $CE(2R_1 + R_2)$ which locates G on BD and F on the given circle. Draw $C(GB)(3C_1 + C_3)$ which cuts FC produced in I. FI is the desired length. Op. $(7R_1 + 4R_2 + 7C_1 + 4C_3)$, simplicity 22, exactness 14, 4 lines, 4 circles.

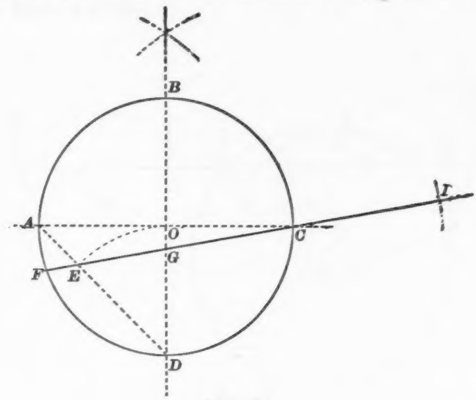

FIG. 1.

2°. *With the square.*—The instrument can only be used as follows: Draw AC with one side of the right angle serving as ruler $(R_1' + R_2)$. Without moving the square, place the ruler against the hypotenuse and slide the square along the ruler until the other side passes through $O(E)$. Draw $BD(R_2)$ and proceed as in the canonical construction above. Op. $(4R_1 + R_1' + 4R_2 + E + 5C_1 + 2C_3)$, simplicity 17, exactness 11, 4 lines, 2 circles.

B. (*Figure* 2.)—This is based on the relation $\sqrt{2} + \sqrt{3} = 3.1462 \cdots$.

1°. *Canonical.*—With any point of the given circle as center draw $B(BO)(C_1 + C_2 + C_3)$ cutting the given circle in C and D. Draw $C(BO)(C_1 + C_3)$ which cuts $B(BO)$ in P. Draw $PO(2R_1 + R_2)$ which cuts the given circle in K. We have $KD = \sqrt{2}$, $CD = \sqrt{3}$. Draw $KD(2R_1 + R_2)$, then $D(DC)(2C_1 + C_3)$ which cuts KD produced in I and I'. KI is the desired approximate length of the semicircumference. Op. $(4R_1 + 2R_2 + 4C_1 + C_2 + 3C_3)$, simplicity 14, exactness 9, 2 lines and 3 circles.

The construction can be performed otherwise and a symbol obtained of the same simplicity,

$$\text{op. } (2R_1 + R_2 + 6C_1 + C_2 + 4C_3)$$

(see *Bulletin de la Société mathématique de France*, volume 23, 1895).

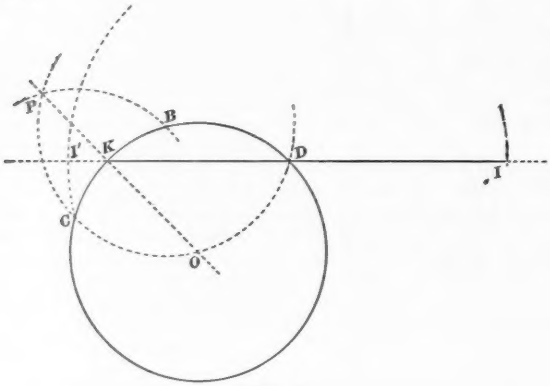

FIG. 2.

Remark.—Since $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 1$, we have approximately $KI' = 1/\pi$, since $\sqrt{3} + \sqrt{2}$ is π.

2°. *With the Square.*—No simplification can be obtained in this case.

C. (*Figure* 3.)—This rests on the fact that the cosine of the smallest positive angle whose sine is the side of the inscribed regular decagon in the circle of radius 1 is

$$\sqrt{\tfrac{1}{2}(\sqrt{5} - 1)} = 0.7861,$$

while

$$\tfrac{1}{4}\pi = 0.78539 \cdots .$$

1°. *Canonical.*—Draw a diameter $AB(R_1 + R_2)$. Draw $A(AB)(2C_1 + C_3)$, which cuts AB in B'. Draw

$$B'(AB)(C_1 + C_3),$$

which cuts $A(AB)$ in C and C'. Draw $C'(AB)(C_1 + C_3)$, which cuts $B'(AB)$ in D. Draw $AD(2R_1 + R_2)$, which cuts

$A(AB)$ in E between A and D. Draw $C(CE)(2C_1 + C_3)$, which cuts AB in G between A and B and in G'. Draw $A(AG)(2C_1 + C_3)$, which cuts the given circle in H. The distance BH, which need not be drawn, is one-quarter of the circumference.

Op. $(3R_1 + 2R_2 + 8C_1 + 5C_3)$, simplicity 18, exactness 11, 2 lines, 5 circles.

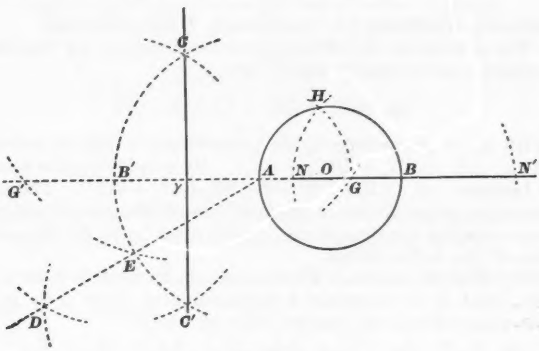

FIG. 3.

In fact it suffices to show that $AG = AH = AB\dfrac{\sqrt{5}-1}{2}$.

For $\frac{1}{2}(\sqrt{5}-1)$ will then be the sine of the angle ABH, and hence the cosine of the same angle will be $\sqrt{\frac{1}{2}(\sqrt{5}-1.)}$ But the isosceles triangle CAE is right-angled at A, hence $CE = AB\sqrt{2}$. If we denote by γ the point where CC' cuts BA, we have, in the equilateral triangle $CB'A$, $C\gamma = \frac{1}{2}AB\sqrt{3}$, whence

$$\overline{CG}^2 = \overline{CE}^2 = \overline{C\gamma}^2 + \overline{\gamma G}^2,$$

or

$$2\overline{AB}^2 = \tfrac{3}{4}\overline{AB}^2 + \overline{\gamma G}^2,$$

or

$$\overline{\gamma G}^2 = \tfrac{5}{4}\overline{AB}^2,$$

$$\therefore\ AG = \gamma G - \gamma A = \tfrac{1}{2}AB(\sqrt{5}-1).$$

Remark.—We should find $AG' = \frac{1}{2}AB(\sqrt{5}+1)$. G and G' therefore divide AB in extreme and mean ratio (sectio aurea). Now in the construction above G and G' are located by the symbol op. $(3R_1 + 2R_2 + 6C_1 + 4C_3)$, but if we merely wish to divide a line AB in extreme and mean ratio, the symbol should be diminished by $(R_1 + R_2)$, since AB need not be drawn. It appears then that to divide a line in extreme and mean ratio, the symbol

$$\text{op. } (2R_1 + R_2 + 6C_1 + 4C_3)$$

suffices; simplicity 13, exactness 8, 1 line, 4 circles.

The symbol of the old classical construction for the same problem, as ordinarily given, is

$$\text{op. } (6R_1 + 3R_2 + 11C_1 + 9C_3).$$

With some geometrographic precautions it can be reduced to op. $(6R_1 + 3R_2 + 10C_1 + 8C_3)$. By employing the square it becomes op. $(4R_1 + 2R_1' + E + 3R_2 + 8C_1 + 6C_3)$. The construction given above is in fact one of the numerous geometrographic constructions* which are known for the problem of the sectio aurea.

2°. *With the square.*—The square can be used to determine the point E, by drawing a perpendicular from A to $B'C'$. But this reduces the symbol only by a unit,

$$\text{op. } (R_1 + 2R_1' + E + 2R_2 + 7C_1 + 4C_3),$$

simplicity 17, exactness 11, 2 lines, 4 circles.

D. (*Figure* 4.)—1°. *Canonical.*—Draw any diameter AB $(R_1 + R_2)$; draw $A(AO)(2C_1 + C_3)$ cutting the given circumference in C and D; draw $CD(2R_1 + R_2)$ cutting AB in E. On AB produced lay off $EF = 2CD(4C_1 + 2C_3)$; draw FD $(2R_1 + R_2)$ and on this line take H, between F and D, so that $FH = AB(3C_1 + C_3)$. HD is a quarter circumference.

In fact, we find

$$HD = \tfrac{1}{2}\sqrt{51} - 2 = 1.570714 \cdots$$

while the quarter circumference is $1.570796 \cdots$.

$$\text{Op. } (5R_1 + 3R_2 + 9C_1 + 4C_3),$$

simplicity 21, exactness 14, 3 lines, 4 circles.

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* A construction for a given problem is called its geometrographic construction if it has the minimum coefficient of simplicity of all the known constructions. If there are several constructions of the minimum simplicity they are all called geometrographic.

2°. *With the square.*—The square is not useful in this construction.

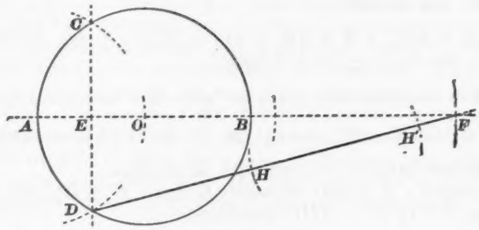

FIG. 4.

SUMMARY.

Let $\triangle$ be the difference between π and its approximation.

A. The symbol for the canonical construction of π is

op. $(7R_1 + 4R_2 + 7C_1 + 4C_3)$; S. 22, E. 14 ; 4 lines, 4 circles.

With the square

op. $(4R_1 + R_1' + 4R_2 + E + 5C_1 + 2C_3)$; S. 17, E. 11 ; 4 lines, 2 circles. $\triangle = +0.0012\cdots$.

B. The symbol for the canonical construction of π is

op. $(4R_1 + 2R_2 + 4C_1 + C_2 + 3C_3)$; S. 14, E. 19 ; 2 lines, 3 circles.

There is no advantage in using the square.

$$\triangle = +0.0047\cdots.$$

C. The construction gives $\frac{1}{2}\pi$ with the canonical symbol

op. $(3R_1 + 2R_2 + 8C_1 + 5C_3)$; S. 18, E. 11 ; 2 lines, 5 circles.

With the square

op. $(R_1 + 2R_1' + E + 2R_2 + 7C_1 + 4C_3)$; S. 17, E. 11 ; 2 lines, 4 circles.

To obtain π it is necessary to draw $B(BH)$ cutting AB in two points N and N' whose distance represents π ; this adds $(2C_1 + C_3)$ to the preceding symbols.

The symbols then become, for the canonical construction, op. $(3R_1 + 2R_2 + 10C_1 + 6C_3)$; S. 21, E. 13; 2 lines, 6 circles.

With the square

op. $(R_1 + 2R_1' + E + 2R_2 + 9C_1 + 5C_3)$; S. 20, E. 13; 2 lines, 5 circles. $\triangle = + 0.0030 \cdots$.

D. The construction gives $\frac{1}{2}\pi$ with the canonical symbol

op. $(5R_1 + 3R_2 + 9C_1 + 4C_3)$; S. 21, E. 14; 3 lines, 4 circles.

The square cannot be used with advantage.

To obtain π it is still necessary to draw $H(HD)(2C_1 + C_3)$ cutting FD in H'. HH' represents π.

Op. $(5R_1 + 3R_2 + 11C_1 + 5C_3)$; S. 24, E. 16; 3 lines, 5 circles. $\triangle = -0.0002$.

Remark.—If the required length is $\frac{1}{2}\pi$, it is necessary in the construction A to bisect FI. in the construction B to bisect KI. This requires op. $(2R_1 + R_2 + 2C_1 + 2C_3)$. The simplicities of the canonical constructions A, B, C, D for $\frac{1}{2}\pi$ are therefore 29, 21, 18, 21, respectively.*

The geometrographic symbol thus affords the greatest facility for the comparison of constructions and the choice of the simplest or the one best adapted for the purpose.

Trisection of an Angle.

I take advantage of this opportunity to mention an approximate trisection of a given angle, which is very little known. This construction was communicated to me by Carl Störmer, a young Norwegian mathematician, who got it from a sea-captain evidently not well versed in mathematics, since he claimed to have discovered an exact trisection by means of the ruler and compass.

In Fig. 5, let $AOB = \alpha < \frac{1}{2}\pi$ be the given angle. "Draw any circle with center at O, cutting OA in A, OB in B. Join the middle point C of OB to the end A' of the diameter OA. Draw CA' cutting the diameter perpendicular to OA in D. Through D draw a parallel to AA' cutting the given circle in E on the side of A'. Draw EB, and parallels to EB through A' and O. These two parallels cut the circumfer-

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* Mr. E. B. Escott calls my attention to two other approximate constructions for π, one given by R. A. Proctor in his Light science for leisure hours, Vol. I, the other by A. A. Kochanski, in the *Acta eruditorum* for 1685 (comp. Cantor's Geschichte, Vol. III, p. 21). For the former, $\triangle = 0.0002$; for the latter, $\triangle = 0.00006$.

ence in F and G between A and B so that arc $FG = \frac{1}{3}$ arc BA."

This is the enunciation as communicated to me. I will now treat the resulting construction geometrographically.

1°. *Geometrographic canonical construction.*—Fig. 5, $a < 90°$. Observe first that in the preceding construction, if only the value of $\frac{1}{3}a$ is required, we may stop after drawing DE and draw OE. The angle $A'OE$ will be $\frac{1}{3}a$ to the degree of approximation obtainable by the construction. In fact OG, parallel to $A'F$, divides the angle AOF rigorously into two equal parts, and since, from the above, arc $GF = \frac{1}{3}a$, it follows that the arcs GA and its equal $A'E$ are also each $= \frac{1}{3}a$.

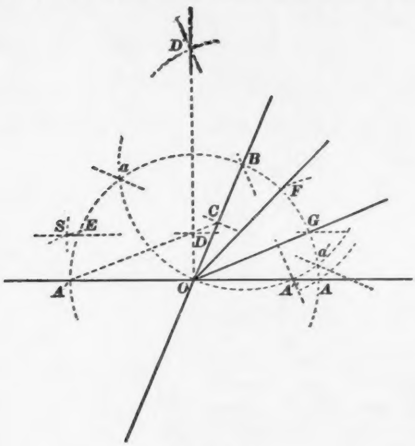

FIG. 5.

With any radius ρ, draw $O(\rho)(C_1 + C_3)$ cutting OA in A, OB in B. Draw $B(\rho)(C_1 + C_3)$ intersecting $O(\rho)$ in a, a'; connect a and $a'(2R_1 + R_2)$, thus locating C on OB. Let A'' be the point where $B(\rho)$ cuts OA. Draw $BA''(2R_1 + R_2)$ cutting $B(\rho)$ in D'. Draw $D'O(2R_1 + R_2)$, which is the perpendicular to OA at O. Draw $CA'(2R_1 + R_2)$ locating D on OD'. To pass a parallel to OA through D, draw $D(\rho)(C_1 + C_3)$ and, having taken DO on the compass while the one point was at D in drawing $D(\rho)$, draw $A'(DO)$ $(2C_1 + C_3)$ cutting $D(\rho)$ in S outside of $O(\rho)$. Draw DS $(2R_1 + R_2)$ cutting $O(\rho)$ in E on the side of A' and in G on the side of A (for arc $A'E = \frac{1}{3}\alpha = AG$). Draw $OG(2R_1 + R_2)$; then $AOG = \frac{1}{3}AOB$ will be obtained by

$$\text{op. } (12R_1 + 6R_2 + 5C_1 + 4C_3),$$

simplicity 27, exactness 17 ; 6 lines, 4 circles. To trisect completely the angle AOB, we must draw also $G(GA)$ $(2C_1 + C_3)$ cutting $O(\rho)$ in F, and finally $OF(2R_1 + R_2)$. In all,

$$\text{op. } (14R_1 + 7R_2 + 7C_1 + 5C_3),$$

simplicity 33, exactness 21 ; 7 lines, 5 circles.

2°. *With the square.*—Draw $O(\rho)$, $B(\rho)$, and their intersections, $(2R_1 + R_2 + 2C_1 + 2C_3)$. Draw $CA'(2R_1 + R_2)$ and the perpendicular OD' to $OA(2R_1' + E + R_2)$. Draw the $(2R_1 + R_2)$ parallel through D to $OA(2R_1' + E + R_2)$, and GO

$$\text{op. } (6R_1 + 4R_1' + 2E + 5R_2 + 2C_1 + 2C_3),$$

S. 21, E. 14 ; 5 lines, 2 circles. To trisect AOB completely,

$$\text{op. } (8R_1 + 4R_1' + 2E + 6R_2 + 4C_1 + 3C_3),$$

S. 27, E. 18; 6 lines, 3 circles. M. Störmer informs me that on following out the construction he finds

$$\measuredangle FOG = \tfrac{1}{3}\alpha = \sin^{-1}\frac{\sin\alpha}{2 + \cos\alpha}$$

and that the maximum error is less than $20'$.

For $\alpha > 90°$ the *graphical* operation is somewhat more complicated since it is necessary to operate on $\alpha - 90°$.

1°. *Canonical*, Fig. 6.—Erect at O a perpendicular OB' to OB by drawing any circle $\omega(\rho_1)$ passing through $O(C_1 + C_3)$. If this circle cuts OB in B_1, draw $B_1\omega(2R_1 + R_2)$ cutting $\omega(\rho_1)$ in δ; then $O\delta(2R_1 + R_2)$ is the required perpendicular to OB. Taking B' in the proper sense, we operate on the angle AOB' as formerly on AOB, at least with very slight modification. Draw $O(\rho)(C_1 + C_3)$ locating A and B', then $B'(\rho)(C_1 + C_3)$ locating a, a', and A'', then $aa'(2R_1 + R_2)$ locating C in the middle of OB'. Draw also $a(\rho)(C_1 + C_3)$, which will be of use presently. $a(\rho)$ cuts aa' in β on the same side of OB' as A. Draw $B'A''(2R_1 + R_2)$, which locates D'. Draw $OD'(2R_1 + R_2)$ and $CA'(2R_1 + R_2)$, which locates D. Fix the point $S(3C_1 + 2C_3)$ as in the case $\alpha < 90°$ and draw $SD(2R_1 + R_2)$ locating G' on $O(\rho)$. We have then arc $AG' = \frac{1}{3}(\alpha - \frac{1}{2}\pi)$. Draw $G'(O\beta)(3C_1 + C_3)$ locating G on $O(\rho)$ so that arc $AG'G = \frac{1}{3}\alpha$, since

$$\text{arc } GG' = \text{arc } O\beta = \text{arc } 30^\circ = \text{arc } \tfrac{1}{6}\pi.$$

Draw $OG(2R_1 + R_2)$, and we have $AOG = \frac{1}{3}a$.

$$\text{op. } (16R_1 + 8R_2 + 10C_1 + 7C_3),$$

S. 41, E. 26; 8 lines, 7 circles.

To trisect AOB completely we must still draw $G(GA)$ $(2C_1 + C_3)$ locating F on $O(\rho)$ and $OF(2R_1 + R_2)$,

$$\text{op. } (18R_1 + 9R_2 + 12C_1 + 8C_3),$$

S. 47, E. 30; 9 lines, 8 circles.

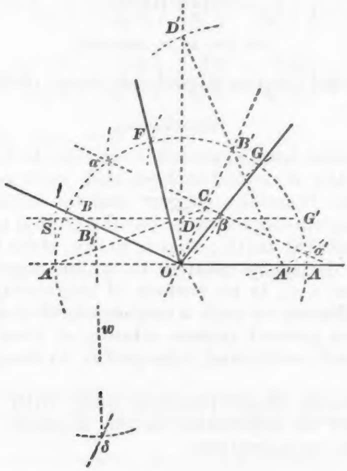

FIG. 6.

2°. *With the square.*—We find for constructing $GOA = \frac{1}{3}a$, op. $(6R_1 + 6R_1' + 3E + 6R_2 + 6C_1 + 4C_3)$, S. 31, E. 21; 6 lines, 4 circles. For trisecting AOB completely,

$$\text{op. } (8R_1 + 6R_1' + 3E + 7R_2 + 8C_1 + 5C_3),$$

S. 37, E. 25; 7 lines, 5 circles.

NOTE.—Since geometrography is a very new science and therefore little known to many mathematicians, I have developed the constructions in great detail in order that they may be followed readily by all whom this paper may interest.

Geometrography is treated didactically in the Traité de géométrie of Rouché et de Comberousse (7th edition, volume 1, Gauthier-Villars, Paris, 1900), in the *Archiv der Mathematik und Physik*, April and May, 1901, and more fully in my La géométrografie, Paris, Naud, in press, 8vo. 100 pp.

CONCERNING THE ELLIPTIC $\wp(g_2, g_3, z)$-FUNCTIONS AS COÖRDINATES IN A LINE COMPLEX, AND CERTAIN RELATED THEOREMS.

BY DR. H. F. STECKER.

(Read before the American Mathematical Society, October 26, 1901.)

Introduction.

SYSTEMS, that have appeared from time to time, of coördinates for the Kummer surface, each more or less related to the elliptic functions, suggest that the existence of such systems of coördinates may be but the partial manifestation of a more general truth; that is to say, since the Kummer surface is definitely related to a line complex of the second order, *i. e.*, is its surface of singularities, any system of coördinates on such a surface ought to arrange itself under a more general system relating at least to the complex of second order, and presumably to the general complex.

The following paper concerns itself with this general question and its application to the Kummer surface and certain other configurations.

§ I.

If we write the general quartic which enters into the discussion of the elliptic functions in the form

$$F(z) \equiv z^4 + \alpha z^3 + \beta z^2 + \gamma z + \delta \equiv \prod_{1,2,3,4.} (z_2^{(\kappa)} z_1 - z_1^{(\kappa)} z_2) = 0,$$

and if

$$(i, \kappa) \equiv \begin{vmatrix} Z_1^{(i)} & Z_2^{(i)} \\ Z_1^{(\kappa)} & Z_2^{(\kappa)} \end{vmatrix},$$

then $F(z)$ has the following irrational invariants:

$$(1, 2)\,(3, 4) \equiv R, \quad (1, 3)\,(4, 2) \equiv S, \quad (1, 4)\,(2, 3) \equiv T,$$

where $R + S + T = 0$.

Put $$A \equiv \frac{R}{6} - \frac{S}{6}, \quad B \equiv \frac{S}{6} - \frac{T}{6}, \quad C \equiv \frac{T}{6} - \frac{R}{6},$$

where again $A + B + C = 0$;

write $$A \equiv \left(\sqrt{\frac{R}{6}} + \sqrt{\frac{S}{6}}\right)\left(\sqrt{\frac{R}{6}} - \sqrt{\frac{S}{6}}\right) \equiv n_1 n_4,$$

$$B \equiv \left(\sqrt{\frac{S}{6}} + \sqrt{\frac{T}{6}}\right)\left(\sqrt{\frac{S}{6}} - \sqrt{\frac{T}{6}}\right) \equiv n_2 n_5,$$

$$C \equiv \left(\sqrt{\frac{T}{6}} + \sqrt{\frac{R}{6}}\right)\left(\sqrt{\frac{T}{6}} - \sqrt{\frac{R}{6}}\right) \equiv n_3 n_6;$$

then $n_1 n_4 + n_2 n_5 + n_3 n_6 = 0$, or say

$$\sum_{1,2,3} n_\lambda n_{\lambda+3} = 0.$$

Then, since $-g_2 \equiv AB + AC + BC$ and $2g_3 \equiv ABC$,

$$-g_2 \equiv n_1 n_2 n_4 n_5 + n_1 n_3 n_4 n_6 + n_2 n_3 n_5 n_6,$$

$$2\,g_3 \equiv n_1 n_2 n_3 n_4 n_5 n_6.$$

§ II.

We may now consider the n_i's either as (a) line-coördinates themselves or (b) as Klein's fundamental complexes.

(a) We put $n_i \equiv p_i$.

Then we have in the first place the two complexes

(1) $$g_2 = -\prod^{\kappa=1,\ldots,6}\left\{p_\kappa \sum^{\lambda=1,2,3} \frac{1}{p_\lambda p_{\lambda+3}}\right\},$$

(2) $$g_3 = \tfrac{1}{2}\prod^{1,\ldots,6} p_\kappa;$$

write also (3) $x = p_5$, (4) $y = p_6$,

where $z \equiv x + iy$ and hence $z \equiv p_5 + ip_6$.

Together with these four relations we have the identical relation

$$(5)\qquad I \equiv p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

Suppose then that we have any complex given by an equation

$$(6)\qquad \Omega_n(p_1, p_2, \cdots, p_6) = 0.$$

Consider any definite line in the complex whose coördinates are $P_i \equiv a_i \; (i = 1, \cdots, 6)$. Substituting these values in equations (1), $\cdots$, (4), we have

$$g_2 = -\prod^{1,\ldots 6}\left\{a_\kappa \sum^{1,2,3} \frac{1}{a_\lambda a_{\lambda+3}}\right\},$$

$$g_3 = \tfrac{1}{2}\prod^{1\ldots 6} a_\kappa, \quad z = a_5 + i a_6,$$

that is, definite numerical values for g_2, g_3 and z, and hence a definite numerical value for

$$\wp(g_2, g_3, z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \frac{g_2^{\,2}}{1200} z^6 + \cdots,$$

i. e., to that line corresponds

$$(7)\qquad \wp\left[-\prod^{1\ldots 6}\left\{a_\kappa \sum^{1,2,3} \frac{1}{a_\lambda a_{\lambda+3}}\right\}, \tfrac{1}{2}\prod^{1\ldots 6} a_\lambda,\ a_5 + i a_6\right].$$

Conversely, given a definite $\wp$-function, from it we can determine g_2, g_3, and z, numerically, say

$$g_2 = h_1, \quad g_3 = h_2, \quad z = h_3 + i h_4.$$

Substituting these values in equations (1), $\cdots$, (5), we have

$$(8)\qquad -\prod^{1\cdots 6}\left\{p_\kappa \sum^{1\cdots 3} \frac{1}{p_\lambda p_{\lambda+3}}\right\} = h_1,$$

$$(9)\qquad \tfrac{1}{2}\prod^{1\cdots 6} p_\kappa = h_2,$$

$$(10)\qquad p_5 = h_3, \qquad (11)\qquad p_6 = h_4,$$

$$(12)\qquad p_1 p_4 + p_2 p_5 + p_3 p_6 = 0.$$

These together with the equation of the complex

$$(6)\qquad \Omega_n(p_1, \cdots, p_6) = 0$$

make six equations from which to determine the six quantities p_κ ($\kappa = 1, \cdots, 6$) in terms of h_i and the coefficients of (6).

If $p_6 = 0$ then no other p_i can vanish, so that the *$\wp$-function is not infinite for any real line of the complex.* If we put $p_6 = ip_5$ then the $\wp$-function is infinite and we may say that the *infinite values of the $\wp$-function correspond to a definite set of imaginary lines.*

For lines of the complex with finite coördinates we have $p_6^2 \neq 0$. Then if any other p_κ ($\kappa = 1, \cdots, 5$) vanish we have

$$
(13)\quad
\begin{aligned}
&g_3 = 0,\\
&g_2 = -p_s p_{s+3} p_t p_{t+3},\\
&I \equiv p_s p_{s+3} + p_t p_{t+3} = 0
\end{aligned}
\qquad
\begin{aligned}
&(s,\ t = 1, \cdots, 5\ ;\ s \neq t,\ \kappa\ ;\\
&t \neq \kappa,\ s + 3).
\end{aligned}
$$

From these we find

$$p_t p_{t+3} = \sqrt{g_2}, \quad p_s p_{s+3} = -\sqrt{g_2},$$

so that the equation of the complex takes the form

$$(14)\qquad F\left(\frac{\sqrt{g_2}}{p_t},\ -\frac{\sqrt{g_2}}{p_s},\ p_r,\ p_s,\ p_t\right) = 0,$$

where p_r is the coefficient of p_κ in the identity. Hence:

The vanishing of g_3 characterizes the ruled surface

$$F\left(\frac{\sqrt{g_2}}{p_t},\ -\frac{\sqrt{g_2}}{p_s},\ p_t,\ p_s,\ p_r\right) = 0.$$

The corresponding $\wp$-function is $\wp\,(p_t^2 p_{t+3}^2,\ 0,\ p_5 + ip_6)$.

If g_2 vanish, we have

$$p_1 p_2 p_4 p_5 = -\,(p_1 p_3 p_4 p_6 + p_2 p_3 p_5 p_6).$$

From (2) the left member is equal to $\dfrac{2g_3}{p_3 p_6}$, which gives

$$(p_3 p_6)^2 = \frac{-\,2g_3}{p_1 p_4 + p_2 p_5},$$

or, making use of the identical relation $I = 0$,

$$p_3 p_6 = \sqrt[3]{2g_3}\ ;$$

with this value we find

$$p_1 p_4 = \omega^2 \sqrt[3]{2g_3}, \quad p_2 p_5 = \omega \sqrt[3]{2g_3},$$

where ω is an imaginary cube root of unity, so that for $g_2 = 0$ the identical relation is simply

$$\omega^2 + \omega + 1 = 0.$$

The equation of the complex becomes then, *when* $g_2 = 0$,

$$\Omega_n \left[\omega^2 \frac{\sqrt[3]{2g_3}}{p_4}, \ \omega \frac{\sqrt[3]{2g_3}}{p_5}, \ \frac{\sqrt[3]{2g_3}}{p_6}, \ p_4, \ p_5, \ p_6 \right] = 0$$

which is a ruled surface; the identical relation becomes

$$\omega^2 + \omega + 1 = 0.$$

The corresponding $\wp$*-function is*

$$\wp\{0, \tfrac{1}{2} p_1^2 p_4^2, p_5 + i p_6.\}$$

§ III.

Consider next Klein's fundamental complexes and put

$$A = x_1^2 + x_4^2; \quad B = x_2^2 + x_5^2; \quad C = x_3^2 + x_6^2,$$

so that

$$\sum^{1 \dots 6} x_i^2 = 0.$$

Then we have

15 $$g_3 = \tfrac{1}{2} \prod^{1,2,3} (x_i^2 + x_{i+3}^2), \qquad g_2 = \sum^{1,2,3} \frac{-g_3}{x_i^2 + x_{i+3}^2}$$

and in the value of z we write $x = x_4$ and $y = x_6$, so that $z = x_4 + i x_6$.

Then, as in the first case, we shall have always six equations to determine the x's; these in turn give six equations to determine the p's, or true coördinates.

For a directrix of the congruence determined by the two complexes x_1 and x_2, say the congruence $(x_1 x_2)$ we have $x_1^2 + x_2^2 = 0$, that is, an edge of a definite fundamental tetrahedron. Each of the three factors of g_3 corresponds to two, and hence all three factors to the six edges of this tetrahedron. *Hence the tetrahedron formed by the directrices of congruences* $(x_1 x_2)$, $(x_3 x_4)$, *and* $(x_5 x_6)$ *is characterized by* $g_3 = 0$. *The corresponding* $\wp$*-functions are* $\wp(0, 0, \infty)$; $\wp(0, 0, \pm 1)$.

But the particular interest which attaches to these fundamental complexes is their easy application to the complex of second order, viz., the equation then—as Klein shows—is simply $\sum \kappa_i x_i^2 = 0$. Also

$$x_i^2 = \frac{\prod\limits^{p=1,2,3,4} (x_i - \lambda_p)}{\rho f'(x_i)}, \qquad (i = 1, \cdots, 6),$$

where

$$f(\lambda) \equiv \prod (x_i - \lambda)$$

and the λ_p are roots of

$$\sum_{1 \cdots 6} \frac{x_i^2}{x_i - \lambda} = 0.*$$

Then we have

$$x_\alpha^2 + x_\beta^2 = \frac{1}{\rho f'(x_\alpha) f'(x_\beta)} \begin{vmatrix} f'(x_\beta) \prod (x_\beta - \lambda_p) \\ f'(x_\alpha) \prod (x_\alpha - \lambda_p) \end{vmatrix}$$

or say

$$\frac{\Delta_{\alpha\beta}}{\rho f'(x_\alpha) f'(x_\beta)};$$

whence

$$g_3 = \frac{1}{2\rho^3} \prod^{1,2,3} \frac{\Delta_{i,i+3}}{f'(x_i) f'(x_{i+3})}, \quad g_2 = \rho \sum \frac{-g_3 f'(x_i) f'(x_{i+3})}{\Delta_{i,i+3}}.$$

For the tangent to the Kummer surface two λ's are equal and for the 16 double planes or double points all four are equal.

Write β as the value of the equal λ's. Also put

$$P_{s,t}(m,n,r) \equiv \begin{vmatrix} (x_s - \beta)^m i \left[\dfrac{[x_t - \lambda_1 \cdot x_t - \lambda_2]^n}{\rho f'(x_t)} \right]^r \\ (x_t - \beta)^m \left[\dfrac{[x_s - \lambda_1 \cdot x_s - \lambda_2]^n}{\rho f'(x_s)} \right]^r \end{vmatrix}$$

Then the Kummer surface is characterized by

$$\varphi \left\{ \sum \frac{\Pi P_{s,s+3}(2,1,1)}{P_{i,i+3}(2,1,1)}, \Pi P_{s,s+3}(2,1,1), P_{4,6}(1,1,\tfrac{1}{2}) \right\}$$

and its 16 double planes and double points by

$$\varphi \left\{ \sum \frac{\Pi P_{s,s+3}(401)}{P_{i,i+3}(401)}, \Pi P_{s,s+3}(401), P_{4,6}(2,0,\tfrac{1}{2}) \right\}$$

CORNELL UNIVERSITY,
October 12, 1901.

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* Klein, *Math. Annalen*, vol. 5, pp. 294–5.

ON THE ABELIAN GROUPS WHICH ARE CONFORMAL WITH NON-ABELIAN GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, October 26, 1901.)

Two distinct groups are said to be conformal when they contain the same number of operators of each order.* The present paper is devoted to the determination of all the abelian groups which are conformal with non-abelian groups. The complete solution of the converse of this problem, *viz.*, the determination of all the non-abelian groups which are conformal with abelian ones is much more difficult, since a large number of distinct non-abelian groups may be conformal with the same abelian group while no more than one abelian group can be conformal with one non-abelian group. In fact, two distinct abelian groups cannot be conformal.

It is well known that there is only one group of order 2^m which does not include any operator of order 4, *viz.*, the group of type $(1, 1, 1, \cdots\cdots)$.† Moreover, there is only one cyclic group of order 2^m, and when $m < 4$ no two groups of order 2^m are conformal. We proceed to prove that every abelian group G of order 2^m which does not satisfy one of these conditions is conformal with at least one non-abelian group.

Let H be the subgroup of G which is generated by the square of one of its independent generators s of lowest order together with all the other independent generators of G. The order of H is 2^{m-1}. Since $m > 3$ there is an operator t of order 2 which has the following properties.‡ It transforms H into itself, it is commutative with half of the operators of H (including all those which are not of highest order), and it transforms the rest into themselves multiplied by an operator of order 2 which is not the square of a non-invariant operator of H; *i. e.*, t does not transform an operator of order 4 contained in H into its inverse. The non-abelian group generated by H and t is conformal with G whenever $s^2 = 1$.

When the order of s exceeds two, we may make the group generated by t and H (written as a regular substitution group) simply isomorphic with itself by writing it in two

* *Quar. Jour. of Math.*, vol. 28 (1896), p. 270.
† *Ibid.*, p. 208.
‡ BULLETIN, vol. 5 (1898), p. 245; also vol. 6 (1899), p. 236.

distinct sets of letters.* If in this intransitive group t is replaced by the continued product of t, the substitution of order two which merely permutes corresponding letters of the two systems of intransitivity, and s^2 in one of the systems of letters there results a transitive group which is conformal with G. That is, any abelian group of order 2^m, $m > 3$, which is neither cyclic nor of type (1, 1, 1, ⋯ (m times)) is conformal with at least one non-abelian group.

It will now be assumed that the order of G is p^m (p being an odd prime number and $m > 2$) and that G is non-cyclic. Let H be the subgroup generated by s^p (s being one of the independent generators of lowest order in G) together with all the other independent generators of G. There is an operator t of order p which transforms H into itself, is commutative with each of its operators contained in a subgroup of order p^{m-2}, and transforms the rest into themselves multiplied by invariant operators of order p. This t and H generate a group conformal with G whenever $s^p = 1$; for if s_1 is any substitution of H that is not commutative with t it is easy to see that $(ts_1)^p = ts_1ts_1 \ldots$ (p times) $= ts_1t^{-1}t^2s_1t^{-2}t^3s_1t^{-3}t^4 \cdots t^{1-p}t^ps_1 = s_1^p$. †

When s^p differs from identity the group generated by H and t, written as a regular group, may be made simply isomorphic with itself $p - 1$ times by writing each substitution in p distinct sets of letters, and t may be replaced by the continued product of t, the substitution of order p which merely permutes the corresponding letters of these systems of intransitivity, and the pth power of s in one of these systems. In the resulting group the pth power of the operators will be the same as those of G taken in the same order and hence t will be conformal with G.‡

If a non-abelian group whose order is not some power of a prime is conformal with an abelian group G, it must be the direct product of its subgroups whose orders are powers of single primes, and hence each of these subgroups is conformal with an abelian group and at least one of these is non abelian. From what precedes it may be observed that *the necessary and sufficient conditions that any abelian group of order* $2^{a_0}p_1^{a_1}p_2^{a_2}\cdots$ ($p_1, p_2, \cdots$ *being distinct odd primes*) *is conformal with at least one non-abelian group are:* 1° *at least one of its subgroups of orders* 2^{a_0}, $p_1^{a_1}$, $p_2^{a_2}$, $\cdots$ *is non-cyclic*; 2° *if the order* $p_\beta^{a_\beta}$ *of this subgroup is odd then* $a_\beta > 2$, *if the order*

* *Quar. Jour. of Math.*, vol. 28 (1896), p. 236.

† *Transactions of the Am. Math. Society*, vol. 2 (1901), p. 262.

‡ *Transactions of the Am. Math. Society*, vol. 2 (1901), p. 264.

is even (2^{a_0}) *then the subgroup must involve operators of order* 4 *and* $a_0 > 3$. Since any number of these factors may be non-abelian, there cannot be an upper limit to the number of non-abelian groups which may be conformal with one abelian group. This fact may be seen in many other ways.

STANFORD UNIVERSITY,
October, 1901.

THE INFINITESIMAL GENERATORS OF CERTAIN PARAMETER GROUPS.

BY DR. S. E. SLOCUM.

(Read before the American Mathematical Society, October 26, 1901.)

BY means of the r independent infinitesimal transformations

$$X_j \equiv \sum_1^n {}_k \xi_{jk}(x_1, \cdots, x_n) \frac{\partial}{\partial x_k} \qquad (j = 1, 2, \cdots, r)$$

we may construct a family of transformations

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

with r essential parameters $a_1, \cdots, a_r$, where $f_i(x, a)$ is defined in the neighborhood of the identical transformation by the series

$$f_i(x, a) \equiv x_i + \sum_1^r {}_j a_j X_j x_i + \frac{1}{2!} \sum_1^r {}_j \sum_1^r {}_k X_j X_k x_i + \cdots$$
$$(i = 1, 2, \cdots, n).$$

The transformations defined by these equations for assigned values of the a's may be denoted by T_a. Let the differential operators X_j ($j = 1, 2, \cdots, r$) satisfy Lie's criterion, that is, let

$$X_j X_k - X_k X_j \equiv \sum_1^r {}_s c_{jks} X_s \qquad (j, k = 1, 2, \cdots, r).$$

Then by Lie's chief theorem, the family of transformations T_a, defined by equations (1), forms a group G.* Conse-

* Continuierliche Gruppen, pp. 390–391.

quently the transformation obtained by the successive application to the manifold $x_1, \cdots, x_n$, in the order named, of the transformations denoted by T_a and T_b respectively, with parameters a and b, will be a transformation of the group G, say T_c; that is to say, we shall have

$$T_b T_a = T_c,$$

where

$$c_k = \varphi_k(a_1, \cdots, a_r, b_1, \cdots, b_r) \quad (k = 1, 2, \cdots, r).$$

If this system of equations is written in the form

$$a_k' = \varphi_k(a_1, \cdots, a_r, \alpha_1, \cdots, \alpha_r) \quad (k = 1, 2, \cdots, r),$$

it can be shown that they define an r-parameter group in the variables a and a', with continuous parameters $\alpha_1, \cdots, \alpha_r$, and also that each transformation of the group is generated by an infinitesimal transformation of the group. The group thus defined is termed the *parameter group* of the given group G.*

On pages 97–103 of the *Proceedings of the American Academy of Arts and Sciences*, volume 36, I have shown that the symbols of the infinitesimal transformations which generate the parameter groups are the same for all groups of the same structure; and in the same pages I have also given a method by which these symbols may be obtained from the structural constants belonging to any given structure. The following is a résumé of the method given in that paper.

Let α denote the differential operator $\alpha = \sum_1^r {}_k a_k X_k$. Then equations (1) may be written in the symbolic form

$$x_i' = e^\alpha x_i \qquad (i = 1, 2, \cdots, n),$$

where e^α denotes the operator

$$e^\alpha f = f + \alpha f + \frac{\alpha^2}{2!} f + \frac{\alpha^3}{3!} f + \cdots$$

and $\alpha^{m+1} f = \alpha(\alpha^m f)$. By making the parameters in equations (1) infinitesimal we obtain an infinitesimal transformation of the family, that is to say, a transformation infinitely near the identical transformation. Let δt denote an infinitesimal, and let γ denote the operator $\gamma = \sum_1^r {}_k c_k X_k$, the

* Transformationsgruppen, vol. 1, pp. 401 *et seq.*

c's being arbitrary parameters. Then the transformation $e^{a+\delta t\gamma}$ is infinitely near the transformation e^{a}. Consequently the transformation obtained by the successive application to the manifold $x_1, \cdots, x_n$, in the order named, of the transformation e^{-a}, inverse to e^{a}, and the transformation $e^{a+\delta t\gamma}$ is an infinitesimal transformation. If we denote its parameters by $\delta tb_1, \cdots, \delta tb_r$ and let $\beta = \sum_1^r{}_k b_k X_k$, we have

$$(2) \qquad e^{-a}e^{a+\delta t\gamma} = e^{\delta t\beta},$$

whence

$$1+\delta t\left\{\gamma - \frac{1}{2!}(a, \gamma) + \frac{1}{3!}[a, (a, \gamma)] - \frac{1}{4!}\{a, [a, (a, \gamma)]\} + \cdots\right\}$$
$$+ \cdots\cdots = 1 + \delta t\beta + \cdots,$$

where (a, γ) denotes the alternant $a\gamma - \gamma a$. Equating coefficients of δt,

$$(3) \quad \beta = \gamma - \frac{1}{2!}(a, \gamma) + \frac{1}{3!}[a, (a, \gamma)] - \frac{1}{4!}\{a, [a, (a, \gamma)]\} + \cdots,$$

whence

$$(4) \qquad b_k = \sum_1^r{}_j P_{kj} c_j \qquad (j = 1, 2, \cdots, r),$$

the P's being power series in $a_1, \cdots, a_r$, convergent for all finite values of $a_1, \cdots, a_r$.

Let Δ denote the determinant of the P's. Then if $\Delta \neq 0$ the a's and b's may be taken arbitrarily and the c's determined by means of equations (4), in which case

$$(5) \qquad c_j = \sum_1^r{}_k \frac{Q_{jk}}{\Delta} b_k \qquad (j = 1, 2, \cdots, r)$$

where Q_{jk} denotes the first minor of Δ relative to P_{kj}. Let the transformation $e^{a+\delta t\gamma}$ be denoted by e^{a_1}, where

$$a_1 = \sum_1^r{}_k a_k^{(1)} X_k,$$

the $a_k^{(1)}$ $(k = 1, 2, \cdots, r)$ being arbitrary parameters. Then

$$(6) \qquad e^{a_1} = e^{a+\delta t\gamma},$$

whence

$$(7) \qquad a_k^{(1)} = a_k + \delta tc_k \qquad (k = 1, 2, \cdots, r).$$

This system of equations defines the infinitesimal transformations which generate the parameter group ; but these are also defined by the equations

$$a_k^{(1)} = a_k + \sum_1^r{}_j \xi_{kj}(a) b_j \delta t \quad (k = 1, 2, \cdots, r). \tag{8}$$

Consequently, from equations (7) and (8),

$$c_k = \sum_1^r{}_j \xi_{kj}(a) b_j = \sum_1^r{}_j \frac{Q_{kj}}{\Delta} b_j. \tag{9}$$

Therefore if $A_1, \cdots, A_r$ denote the symbols of the infinitesimal transformations which generate the parameter group, we have

$$A_j = \sum_1^r{}_k \xi_{jk}(a) \frac{\partial}{\partial a_k} = \sum_1^r{}_k \frac{Q_{jk}}{\Delta} \frac{\partial}{\partial a_k} \qquad (j = 1, 2, \cdots, r).$$

Since the form of Δ and of the Q's depends only on the structural constants, the symbols $A_j (j = 1, 2, \cdots, r)$ will be the same for all groups of the same structure.

To illustrate what precedes, consider the three-parameter structure

$$(X_1, X_2) \equiv 0, \ (X_1, X_3) \equiv X_1, \ (X_2, X_3) \equiv \beta X_2, \ (\beta \neq 0, 1).$$

Equation (3) gives

$$\begin{aligned} b_1X_1 + b_2X_2 + b_3X_3 &= c_1X_1 + c_2X_2 + c_3X_3 \\ &- \frac{1}{2!}\left\{(a_1c_3 - a_3c_1)X_1 + \beta(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{3!}\left\{a_3(a_1c_3 - a_3c_1)X_1 + a_3\beta^2(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{4!}\left\{a_3^2(a_1c_3 - a_3c_1)X_1 + a_3^2\beta^3(a_2c_3 - a_3c_2)X_2\right\} \\ &- \frac{1}{5!}\left\{a_3^3(a_1c_3 - a_3c_1)X_1 + a_3^3\beta^4(a_2c_3 - a_3c_2)X_2\right\} \\ &- \cdots \cdots, \end{aligned}$$

whence

$$\begin{aligned} b_1 &= \frac{c_1}{a_3}(e^{a_3} - 1) - \frac{a_1c_3}{a_3^2}(e^{a_3} - a_3 - 1), \\ b_2 &= \frac{c_2}{a_3\beta}(e^{a_3\beta} - 1) - \frac{a_2c_3}{a_3^2\beta}(e^{a_3\beta} - a_3\beta - 1), \\ b_3 &= c_3. \end{aligned} \tag{10}$$

Consequently,

$$\Delta = \begin{vmatrix} \dfrac{e^{a_3}-1}{a_3} & 0 & -\dfrac{a_1}{a_3^2}(e^{a_3}-a_3-1) \\ 0 & \dfrac{e^{a_3\beta}-1}{a_3\beta} & -\dfrac{a_2}{a_3^2\beta}(e^{a_3\beta}-a_3\beta-1) \\ 0 & 0 & 1 \end{vmatrix}$$

and equations (10) give

$$c_1 = \frac{a_3}{e^{a_3}-1}\left\{b_1 + \frac{a_1}{a_3^2}(e^{a_3}-a_3-1)b_3\right\} \equiv \sum_1^r{}_j\, \xi_{1j}(a)b_j,$$

$$c_2 = \frac{a_3\beta}{e^{a_3\beta}-1}\left\{b_2 + \frac{a_2}{a_3^2\beta}(e^{a_3\beta}-a_3\beta-1)b_3\right\} \equiv \sum_1^r{}_j\, \xi_{2j}(a)b_j,$$

$$c_3 = b_3 \equiv \sum_1^r{}_j\, \xi_{3j}(a)b_j.$$

Therefore the symbols of the infinitesimal transformations which generate the parameter group corresponding to the above structure are

$$A_1 \equiv \frac{a_3}{e^{a_3}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3\beta}{e^{a_3\beta}-1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3}-a_3-1)}{a_3(e^{a_3}-1)}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3\beta}-a_3\beta-1)}{a_3(e^{a_3\beta}-1)}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

In the following table are enumerated all possible types of structure of two-, three- and four-parameter complex groups as given by Lie,* and under each structure are given the symbols of the infinitesimal transformations which generate the parameter group corresponding to that structure, obtained by the method explained above.

Groups With Two Parameters.

Type I. $(X_1, X_2) \equiv X_1.$

The symbols of the infinitesimal transformations which generate the parameter group corresponding to this structure are

$$A_1 \equiv \frac{a_2}{e^{a_2}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_1(e^{a_2}-a_2-1)}{a_2(e^{a_2}-1)}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2}.$$

* Continuierliche Gruppen, pp. 565, 571, 574–589; Transformationsgruppen, vol. 3, pp. 713, 716, 723–730.

Type II. $(X_1, X_2) \equiv 0.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2}.$$

GROUPS WITH THREE PARAMETERS.

Type I. $(X_1, X_2) \equiv X_1, \quad (X_1, X_3) \equiv 2X_2, \quad (X_2, X_3) \equiv X_3,$

$$A_1 \equiv \left\{ -\frac{a_2}{2} + \frac{\varphi(e^\phi - e^{-\phi})}{2(e^\phi + e^{-\phi} - 2)} + \frac{a_1 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_1} + \left\{ -a_3 + \frac{a_2 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_2} + \frac{a_3^2}{\varphi^2}\psi \frac{\partial}{\partial a_3},$$

$$A_2 \equiv \left\{ \frac{a_1}{2} - \frac{a_1 a_2}{2\varphi^2}\psi \right\} \frac{\partial}{\partial a_1} + \left\{ \frac{a_2^2}{\varphi^2} - \frac{2a_1 a_3(e^\phi - e^{-\phi})}{\varphi(e^\phi + e^{-\phi} - 2)} \right\} \frac{\partial}{\partial a_2} + \left\{ -\frac{a_3}{2} - \frac{a_2 a_3}{2\varphi^2}\psi \right\} \frac{\partial}{\partial a_3},$$

$$A_3 \equiv \frac{a_1^2}{\varphi^2}\psi \frac{\partial}{\partial a_1} + \left\{ a_1 + \frac{a_1 a_2}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_2} + \left\{ \frac{a_2}{2} + \frac{\varphi(e^\phi - e^{-\phi})}{2(e^\phi + e^{-\phi} - 2)} + \frac{a_1 a_3}{\varphi^2}\psi \right\} \frac{\partial}{\partial a_3},$$

where

$$\varphi \equiv \sqrt{a_2^2 - 4a_1 a_3}, \qquad \psi \equiv \frac{e^\phi(\varphi - 2) - e^{-\phi}(\varphi + 2) + 4}{e^\phi + e^{-\phi} - 2}.$$

Type II. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv \beta X_2,$

$(\beta \neq 0, 1).$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3 \beta}{e^{a_3\beta} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3\beta} - a_3\beta - 1)}{a_3(e^{a_3\beta} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type III. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_2.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type IV. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_1 + X_2.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1},$$

$$A_2 \equiv -\frac{a_3[e^{a_3}(a_3 - 1) + 1]}{(e^{a_3} - 1)^2} \frac{\partial}{\partial a_1} + \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \left\{ \frac{a_1(e^{a_3} - a_3 - 1) + (a_2 - e^{a_2})[e^{a_3}(1 - a_3) - 1]}{a_3(e^{a_3} - 1)} \right\} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}.$$

Type V. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv 0.$

$$A_1 \equiv \frac{a_3}{e^{a_3} - 1} \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_1(e^{a_3} - a_3 - 1)}{a_3(e^{a_3} - 1)} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3}.$$

Type VI. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv -\frac{a_3}{2} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2}{2} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3}.$$

Type VII. $(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv 0.$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{\partial}{\partial a_3}.$$

Groups With Four Parameters.

A. *Without three-parameter involution group.*

Type I.

$$(X_1, X_2) \equiv X_1, \quad (X_1, X_3) \equiv 2X_2, \quad (X_2, X_3) \equiv X_3,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

The symbols of the infinitesimal transformations which generate the parameter group corresponding to this structure are the same as those given under *Type I* of three-parameter structures with the addition of the symbol

$$A_4 \equiv \frac{\partial}{\partial a_4}.$$

Type II.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$
$$(X_1, X_4) \equiv \beta X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv (\beta - 1)X_3,$$
$$(\beta \neq 1).$$

$$A_1 \equiv \frac{a_4\beta}{e^{a_4\beta} - 1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv \frac{a_3(\beta e^{a_4} - e^{a_4\beta} - \beta + 1)}{(e^{a_4\beta} - 1)(e^{a_4} - 1)(\beta - 1)}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(1 - e^{a_4\beta} - \beta e^{-a_4})}{(e^{a_4\beta} - 1)(e^{a_4(\beta-1)} - 1)}\frac{\partial}{\partial a_1} + \frac{a_4(\beta - 1)}{e^{a_4(\beta-1)} - 1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{1}{a_4(e^{a_4\beta} - 1)}\{a_1(e^{a_4\beta} - a_4\beta - 1)$$
$$+ a_2a_3\beta[(1 - \beta)e^{a_4(\beta-1)} + (\beta - 2)e^{a_4\beta} + e^{a_4}]\}\frac{\partial}{\partial a_1}$$
$$+ \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_2}$$
$$+ \frac{a_3(e^{a_4(\beta-1)} - a_4(\beta - 1) - 1)}{a_4(e^{a_4(\beta-1)} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type III.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$
$$(X_1, X_4) \equiv 2X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 2X_2 + X_3.$$

$$A_1 \equiv \frac{2a_4}{e^{2a_4} - 1}\frac{\partial}{\partial a_1},$$

$$A_2 \equiv -\frac{a_3(e^{2a_4} - 2e^{a_4} + 1)}{(e^{2a_4} - 1)(e^{a_4} - 1)}\frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$

$$A_3 \equiv \left\{ \frac{4a_3[(a_4-1)e^{a_4}+1] + a_2(e^{2a_4} - 2e^{a_4} + 1)}{(e^{2a_4}-1)(e^{a_4}-1)} \right\} \frac{\partial}{\partial a_1}$$
$$- \frac{2a_4(a_4 e^{a_4} - e^{a_4} + 1)}{(e^{a_4}-1)^2} \frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4}-1} \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \{ e^{a_4}[(2-a_4)(e^{2a_4}+1) + 2(a_4 - 2e^{a_4})] \} \frac{\partial}{\partial a_1}$$
$$+ \frac{2a_3(a_4 e^{a_4} - e^{a_4} + 1)}{e^{a_4}-1} \frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4} - a_4 - 1)}{a_4(e^{a_4}-1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type IV.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_1,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{a_4}{e^{a_4}-1} \frac{\partial}{\partial a_1},$$

$$A_2 \equiv - \frac{a_3\left[e^{a_4}(a_4-1) + \dfrac{a_4^2}{2} + 1\right]}{(e^{a_4}-1)^2} \frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4}-1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4}-1)} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \left\{ \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4}-1)} + \frac{a_2 a_3}{e^{a_4}-1} \left[e^{a_4}(a_4-3)\left(e^{a_4} + \frac{a_4}{2}\right) \right.\right.$$
$$\left.\left. + \frac{a_4+2}{2}\left(4e^{a_4}-1\right)\right] \right\} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4}-1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type V.

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv 0, \quad (X_2, X_3) \equiv X_2,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{a_4}{e^{a_4}-1} \frac{\partial}{\partial a_1}, \quad A_2 \equiv \frac{a_3}{e^{a_3}-1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_2(e^{a_3} - a_3 - 1)}{a_3(e^{a_3}-1)} \frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4}-1)} \frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_4}.$$

B. With three-parameter involution group.

Type I.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv \alpha X_1, \quad (X_2, X_4) \equiv \beta X_2, \quad (X_3, X_4) \equiv \gamma X_3,$$

$$(\alpha \neq \beta \neq \gamma).$$

$$A_1 \equiv \frac{a_4 \alpha}{e^{a_4 \alpha} - 1} \frac{\partial}{\partial a_1}, \quad A_2 \equiv \frac{a_4 \beta}{e^{a_4 \beta} - 1} \frac{\partial}{\partial a_2}, \quad A_3 \equiv \frac{a_4 \gamma}{e^{a_4 \gamma} - 1} \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4 \alpha} - a_4 \alpha - 1)}{a_4(e^{a_4 \alpha} - 1)} \frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4 \beta} - a_4 \beta - 1)}{a_4(e^{a_4 \beta} - 1)} \frac{\partial}{\partial a_2}$$

$$+ \frac{a_3(e^{a_4 \gamma} - a_4 \gamma - 1)}{a_4(e^{a_4 \gamma} - 1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type II.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv \alpha X_1, \quad (X_2, X_4) \equiv \beta X_2, \quad (X_3, X_4) \equiv X_2 + \beta X_3,$$

$$(\alpha \neq \beta).$$

$$A_1 \equiv \frac{a_4 \alpha}{e^{a_4 \alpha} - 1} \frac{\partial}{\partial a_1}, \quad A_2 \equiv \frac{a_4 \beta}{e^{a_4 \beta} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4[e^{a_4 \beta}(a_4 \beta - 1) + 1]}{(e^{a_4 \beta} - 1)^2} \frac{\partial}{\partial a_2} + \frac{a_4 \beta}{e^{a_4 \beta} - 1} \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4 \alpha} - a_4 \alpha - 1)}{a_4(e^{a_4 \alpha} - 1)} \frac{\partial}{\partial a_1}$$

$$+ \frac{1}{a_4 \beta(e^{a_4 \beta} - 1)} \Big\{ a_3[e^{a_4 \beta}(a_4 \beta - 1) + 1](2e^{a_4 \beta} - a_4 \beta - 2)$$

$$+ a_2 \beta (e^{a_4 \beta} - a_4 \beta - 1) \Big\} \frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4 \beta} - a_4 \beta - 1)}{a_4(e^{a_4 \beta} - 1)} \frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type III.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_1 + X_2, \quad (X_3, X_4) \equiv X_2 + X_3.$$

$$A_1 \equiv \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_1},$$

$$A_2 \equiv \frac{a_4[e^{a_4}(a_4 - 1) + 1]}{(e^{a_4} - 1)^2} \frac{\partial}{\partial a_1} + \frac{a_4}{e^{a_4} - 1} \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4^2 e^{a_4}}{(e^{a_4}-1)^3}\left[e^{a_4} - \frac{a_4}{2}(e^{a_4}+1) - 1\right]\frac{\partial}{\partial a_1}$$

$$- \frac{a_4[e^{a_4}(a_4-1)+1]}{(e^{a_4}-1)^2}\frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4}-1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{1}{a_4(e^{a_4}-1)}\Big\{a_1(e^{a_4}-a_4-1)$$

$$+ a_2(a_4 e^{a_4} - e^{a_4}+1)(2e^{a_4}-a_4-2)$$

$$+ \frac{a_3}{e^{a_4}-1}\left[(a_4^2 e^{a_4} - 2a_4 e^{a_4} + 2e^{a_4} - 2)\frac{2e^{a_4}-a_4-2}{2}\right.$$

$$\left.\left. + \frac{a_4(a_4e^{a_4}-e^{a_4}+1)^2}{e^{a_4}-1}\right]\right\}\frac{\partial}{\partial a_1}$$

$$+ \frac{1}{a_4(e^{a_4}-1)}\left\{a_2(e^{a_4}-a_4-1) + a_3 a_4(a_4 e^{a_4} - e^{a_4}+1)\right\}\frac{\partial}{\partial a_2}$$

$$+ \frac{a_3(e^{a_4}-a_4-1)}{a_4(e^{a_4}-1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type IIIa.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv X_1, \quad (X_3, X_4) \equiv X_2.$$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv -\frac{a_4}{2}\frac{\partial}{\partial a_1} + \frac{\partial}{\partial a_2},$$

$$A_3 \equiv \frac{a_4^2}{12}\frac{\partial}{\partial a_1} - \frac{a_4}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3},$$

$$A_4 \equiv \tfrac{1}{2}\left(a_2 - \frac{a_3 a_4}{6}\right)\frac{\partial}{\partial a_1} + \frac{a_3}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type IV.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv \alpha X_1, \quad (X_2, X_4) \equiv \alpha X_2, \quad (X_3, X_4) \equiv \gamma X_3,$$

$$(\alpha \neq \gamma).$$

$$A_1 \equiv \frac{a_4 \alpha}{e^{a_4\alpha}-1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4\alpha}{e^{a_4\alpha}-1}\frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{a_4\gamma}{e^{a_4\gamma}-1}\frac{\partial}{\partial a_3},$$

$$A_4 \equiv \frac{a_1(e^{a_4\alpha} - a_4\alpha - 1)}{a_4(e^{a_4\alpha} - 1)}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4\alpha} - a_4\alpha - 1)}{a_4(e^{a_4\alpha} - 1)}\frac{\partial}{\partial a_2}$$
$$+ \frac{a_3(e^{a_4\gamma} - a_4\gamma - 1)}{a_4(e^{a_4\gamma} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type V.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv X_2 + X_3.$$
$$A_1 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2},$$
$$A_3 \equiv -\frac{a_4[e^{a_4}(a_4 - 1) + 1]}{(e^{a_4} - 1)^2}\frac{\partial}{\partial a_2} + \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_3},$$
$$A_4 \equiv \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_1} + \frac{1}{a_4(e^{a_4} - 1)}\{a_2(e^{a_4} - a_4 - 1)$$
$$+ a_3a_4(a_4e^{a_4} - e^{a_4} + 1)\}\frac{\partial}{\partial a_2} + \frac{a_3(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type Va.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$
$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv X_2.$$
$$A_1 \equiv \frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{\partial}{\partial a_2},$$
$$A_3 \equiv -\frac{a_4}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_3}, \qquad A_4 \equiv \frac{a_3}{2}\frac{\partial}{\partial a_2} + \frac{\partial}{\partial a_4}.$$

Type VI.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$
$$(X_1, X_4) \equiv X_1, \quad (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv X_3.$$
$$A_1 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_1}, \qquad A_2 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_2}, \qquad A_3 \equiv \frac{a_4}{e^{a_4} - 1}\frac{\partial}{\partial a_3},$$
$$A_4 \equiv \frac{a_1(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_1} + \frac{a_2(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_2}$$
$$+ \frac{a_3(e^{a_4} - a_4 - 1)}{a_4(e^{a_4} - 1)}\frac{\partial}{\partial a_3} + \frac{\partial}{\partial a_4}.$$

Type VIa.

$$(X_1, X_2) \equiv (X_2, X_3) \equiv (X_3, X_1) \equiv 0,$$

$$(X_1, X_4) \equiv 0, \quad (X_2, X_4) \equiv 0, \quad (X_3, X_4) \equiv 0.$$

$$A_1 \equiv \frac{\partial}{\partial a_1}, \quad A_2 \equiv \frac{\partial}{\partial a_2}, \quad A_3 \equiv \frac{\partial}{\partial a_3}, \quad A_4 \equiv \frac{\partial}{\partial a_4}.$$

UNIVERSITY OF CINCINNATI,
October, 1901.

SHORTER NOTICES.

Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen. Von Dr. LUDWIG SCHLESINGER, ordentlichem Professor an der Universität zu Klausenburg. Leipzig, Göschen, 1900. Pp. viii + 310.

THIS little volume, which forms part of the "Sammlung Schubert" (cf. the BULLETIN for January, 1901, p. 192), gives, we believe, the best introduction which has yet appeared to that important side of the theory of ordinary differential equations in which the points of view are those of the theory of functions of a complex variable. Thus the discussion of the nature of singular points holds a central position in the treatment given. The author has been particularly successful in his choice of topics. He has on the one hand restricted himself to the simpler parts of the subject, more than half the volume being devoted to linear differential equations of the second order, and the remainder to the case of a single equation of the first order. By doing this he has succeeded in avoiding long analytical developments which only confuse a beginner without really teaching him anything. On the other hand the author has treated these simple cases in such a way as to bring out clearly a large number of the most important points of view of the modern theory of differential equations. Some of Dr. Schlesinger's own investigations, to mention only one point, on the Laplacian and Eulerian transformations are here set forth in particularly attractive form, although, of course, only for very special differential equations. The reader can turn to a large treatise for further information on these or other questions if he has once become interested in them, and a great merit of this book is that he will almost inevitably become interested in many of the subjects treated.

We have noticed the absence of only one important subject which would seem to belong in a treatment of this sort. We refer to the theory of the conformal transformation effected by the ratio of two solutions of a homogeneous linear differential equation of the second order (Schwarz's s-function). The volume before us does not, of course, touch those sides of the subject with which Lie's name is connected, nor is any special attention devoted to the theory of the real solutions of differential equations, but in those parts of the theory which he professes to treat the author has achieved more than ordinary success.

MAXIME BÔCHER.

Ἰωάννου Ν. Χατζιδάκι — Εἰσαγωγὴ εἰς τὴν Ἀνωτέραν Ἀλγέβραν. Second edition. Athens, 1898.

THE work of Professor Hatzidakis is a scientific development by Weierstrass's methods of the general principles of arithmetic which form the basis of algebra, of calculus, and of the theory of functions. The natural numbers are defined by the process of counting and in the first chapter is placed the *formal* reasoning by which the associative and commutative laws are deduced from the special cases

$$\beta + \alpha = \alpha + \beta, \tag{1}$$

$$\alpha + (\beta + \gamma) = \alpha + \beta + \gamma. \tag{2}$$

The second chapter deals with fractional numbers, which are regarded as collections of fractional units, and the fraction $\frac{\alpha\beta}{\beta}$ is equal to the natural number α. Then any two numbers of the enlarged system are said to be equal if integral equimultiples of them are equal. Multiplication of fractional numbers is defined according to the distributive law, *i. e.*, 1.41×1.73 is the sum of the products obtained by multiplying every one of the units of the one factor by every unit of the other. The product of two units is defined in like manner so that equals multiplied by equals shall give equals. Thus, since we wish the two products $\frac{10}{10} \times \frac{100}{100}$ and 1×1 to be equal and since

$$\tfrac{10}{10} \times \tfrac{100}{100} = 1000\ (\tfrac{1}{10} \times \tfrac{1}{100}),$$

the product $\frac{1}{10} \times \frac{1}{100}$ is defined to be $\frac{1}{1000}$. From this definition it is easy to deduce the three fundamental identities

$$\beta\alpha = \alpha\beta, \qquad (4) \qquad \alpha(\beta\gamma) = \alpha\beta\gamma, \qquad (5)$$

$$(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta. \qquad (6)$$

A feature worthy of imitation is a short chapter on "Zero as a number," where it is shown that when the fractional system is enlarged by the introduction of zero the laws of multiplication still hold with the exception of the law of equals multiplied by unequals and therefore that division by zero is impossible. After this the system of rational numbers is completed by defining the negative numbers (as assemblages of negative units) and establishing the five fundamental identities for the enlarged system.

In the system of rational numbers many problems admit no solution, *e. g.*, the equation $x^2 = 2$. The attempt to solve this by the algorithm for square root determines an unlimited succession of digits $1, c_1, c_2, \ldots$ such that

$$\left(1 + \frac{c_1}{10} + \cdots + \frac{c_\nu}{10^\nu}\right)^2 < 2 < \left(1 + \frac{c_1}{10} + \cdots + \frac{c_\nu + 1}{10^\nu}\right)^2$$
$$(\nu = 1, 2, \cdots).$$

But such sequences also arise in dealing with certain problems which admit rational solutions, *e. g.*, the equation $3x = 4$. Here again the algorithm for division by the decimal scale gives an unlimited succession of digits. In each case we have an assemblage in which the number of units of each kind is definite, although the number of different units is unlimited, and in each case, however many units of the assemblage are taken, the aggregate is always less than 2. The repeating decimal is naturally regarded as defining a number. Further there exists a rational number $\frac{4}{3}$ such that whatever part M of the infinite decimal is taken, $\frac{4}{3}$ can be separated into two parts one of which is greater than M and, whatever part N is taken out of $\frac{4}{3}$. a part of the decimal can be found greater than N. Accordingly the number represented by 1.333... is said to be *equal* to $\frac{4}{3}$.

On the above considerations is based the following extension of the idea of number. Given any determinate assemblage (πλῆθος) of units such that, however many of them we take, the aggregate is always less than some integer, the *totality* (σύνολον) of the units of the assemblage shall be called a number. This is not of itself a satisfactory definition of an irrational number, but what follows can be easily used to supply the deficiency; for we find next the rules for comparison and combination of these new objects of thought. Two numbers are equal when and only when every *part* of each is contained in the other. The sum of two numbers is the totality of all their units. Their product is the totality obtained by multiplying every unit of the one by every unit of the other. From these definitions the associative, commutative, and distributive laws follow at once.

Thus the system of real numbers is completed, and the next chapter deals with the common complex numbers along similar lines. In fact the same words (τὸ σύνολον πολλῶν μονάδων) are used for the definition of every kind of number. A short chapter at the end of the book is devoted to proving that the system of common complex numbers is the only one in which all the ordinary laws of the four species hold, and more general systems are considered in two appendices.

The intervening chapters, however, contain matter of more vital interest in ordinary analysis. On the definition of irrational numbers in Chapter V can be based the treatment of limits and the convergence of infinite processes. In Chapter VII, "On limits," it is proved that if a positive variable constantly increases but always remains less than some natural number it approaches a limit and therefore that $\lim_{\nu=\infty}\left(1+\frac{1}{\nu}\right)^{\nu}$ exists. In Chapter VIII it is shown that e^x is a continuous function of x, and the exponential a^x when x is irrational is defined as $e^{x \log a}$. Chapter IX is an introduction to the theory of equations comprehending a proof of the fundamental proposition of algebra. Chapter X is devoted to determinants and the solution of simultaneous linear equations.

The above topics are treated in an elementary manner, but it is elementary work which breathes the spirit of the higher analysis, which appeals to the æsthetic faculties by the simplicity and uniformity of its methods no less than to the logical faculty by rigor and conciseness. The work exhibits throughout the clearness which is characteristic of Greek writing and proves the vitality and power of the language of Aristotle and Euclid.

A. B. Frizell.

NOTES.

The opening (January) number of Volume III. of the *Transactions* of the American Mathematical Society contains the following papers: "On a class of automorphic functions," by J. I. Hutchinson; "Concerning the existence of surfaces capable of conformal representation upon the plane in such a manner that geodetic lines are represented by a prescribed system of curves," by H. F. Stecker; "Zur Erklärung der Bogenlänge und des Inhaltes einer krummen Fläche," by O. Stolz; "The groups of Steiner in problems of contact," by L. E. Dickson; "Quaternion space," by A. S. Hathaway; "Reciprocal systems of linear differential equations," by E. J. Wilczynski; "On the invariants of quadratic differential forms," by C. N. Haskins; "The second variation of a definite integral when one end-point is variable," by G. A. Bliss; "On the nature and use of the functions employed in the recognition of quadratic residues," by E. McClintock; "A determination of the number of real and imaginary roots of the hypergeometric series," by E. B. Van Vleck; "On the projective axioms of geometry," by E. H. Moore.

The first meeting of the Berlin mathematical society was held on October 31, 1901. Papers were presented by Weingarten, Kneser, Jahnke, Lampe, Landau, Kötter, Heun, and Hermes. Professor J. Weingarten is president, and Professor A. Kneser and Dr. E. Jahnke the secretaries of the society, which has at present about sixty members. Meetings of the society are held monthly. At the meeting of November 27 the following papers were read: Professor E. Lampe: "Ueber eine Frage des geometrischen Mittelnerthes;" Dr. E. Jahnke: "Ueber Lemoine's Bestimmung der Axenrichtungen eines Kegelschnitts;" Dr. E. Landau: "Ueber den casus irreductibilis bei kubischen Gleichungen."

The preliminary programme of the December meeting of the London mathematical society announced papers by Dr. E. W. Hobson, on "Non-uniform convergence and the integration of series," and by Mr. J. H. Michell, on the "Flexure of a circular plate."

Among the papers announced for the December meeting of the Royal astronomical society of London was one descriptive of the manuscripts of Adams on the perturbations of Uranus, by Mr. R. A. Sampson.

At the meeting of the American physical society on December 27, 1901, the following officers were elected: President, Professor A. A. Michelson; vice-president, Professor A. G. Webster; secretary, Professor Ernest Merritt; treasurer, Professor William Hallock; members of the Council, Messrs. Carl Barus, D. B. Brace, and A. L. Kimball.

The Société Mathématique de France elected the following officers at its annual meeting in January: President, L. Raffy; vice-presidents, E. Blutel, E. Borel, E. Carvallo, and P. Painlevé; secretaries, R. Bricard and E. Duporcq; vice secretaries, A. Grévy and M. Servant; treasurer, M. Claude-Lafontaine; archivist, Ch. Bioche; members of the council, E. Goursat, C. A. Laisant, Lucien Lévy and M. d'Ocagne; honorary members of the bureau, P. Appell, L. Cremona, G. Darboux, J. N. Hâton de la Goupillière, G. Humbert, C. Jordan, E. Picard and H. Poincaré.

A new Italian mathematical periodical, *Il Bolletino di Matematica*, is announced to appear under the editorial direction of Professor Alberto Conti, of the Royal normal school at Bologna. The *Bolletino*, which will be issued bimonthly, will represent the mathematical interests of the secondary schools.

Printed sheets, designed by Dr. W. A. Granville, for plotting polar coördinates have recently been placed in the market.

University of Oxford.—The following courses of lectures in mathematical subjects have been announced for Hilary term 1902, each course consisting of two lectures per week, unless otherwise indicated:—By Professor W. Esson: Synthetic geometry of conics; Synthetic geometry of cubics, one hour.—By Professor E. B. Elliott: Elements of elliptic functions; Supplementary lectures on quantics, one hour.—By Professor A. E. H. Love: Theory of potential and electrostatics, three hours.—By Mr. A. L. Dixon: Calculus of finite differences, one hour.—By Mr. J. E. Campbell: Algebra of quantics, one hour.—By Mr. P. J. Kirby: Solid geometry.—By Mr. E. H. Hayes: Statics (continued), one hour; Elementary mechanics.—By Mr. C. H. Thompson: Dynamics of a particle.—By Mr. C. E. Haselfoot: Physical optics.—By Mr. H. T. Gerrans: Hydrodynamics.—By Mr. J. W. Russell: Pure geometry.—By Mr. C. Leudesdorf: Geometry (maxima and minima, inversion, etc.).—By Mr. C. H. Sampson: Analytical geometry (continued).

University of Paris.—The following courses in mathematical subjects are announced by the faculty of sciences for the current semester, each course consisting of two lectures per week:—By Professor G. Darboux: The general principles of infinitesimal geometry.—By Professor E. Goursat: The operations of the differential and integral calculus and their geometric applications.—By Professor P. Appell: The general laws of equilibrium and motion.—By Professor H. Poincaré: The calculation of the perturbations of the small planets and in particular the methods of Gyldén and Hansen.—By Professor J. Boussinesq: The mechanical theory of light.—By Professor G. Koenigs: Kinematics of solid and deformable bodies, with application to the study of machines.—By Professor E. Bouty: Optics—By Professor H. Pellat: Acoustics and thermodynamics.—By Professor L. Raffy: Mathematical theories introductory to courses in science (notions of analytical geometry, derivatives and integrals, differential equations, general laws of equilibrium, motion of points and systems).—By Dr. M. Andoyer: Elementary theory of planetary motion.

Conférences are conducted by Professors Raffy, J. Hadamard, and P. Puiseux on higher geometry, infinitesimal calculus, and mechanics, respectively, and by Dr. Andoyer, Professor Hadamard and Dr. E. Blutel preparatory to the agrégation.

The preliminary announcements for the second semester of the current academic year include the following courses:—By Professor E. Picard: Theory of functions of two complex variables, and in particular, algebraic, hyperfuchsian, and hyperabelian functions.—By Professor E. Goursat: Differential equations.—By Professor P. Appell: General laws of motion of systems, analytical mechanics, hydrostatics and hydrodynamics.—By Professor G. Koenigs: Theory of machines.—By Professor L. Raffy: Differential equations and their applications to mechanics and physics.

College of France.—The following mathematical courses are given during the first semester of the present academic year, opening December 2, 1901:—By Professor Camille Jordan: Analysis of the works of Hermite, two hours.—By Professor J. Hadamard: The calculus of variations, two hours.

The prize offered by the Göttingen academy of sciences for a memoir developing the law of reciprocity of the lth power residue for an arbitrary numerical body where l is an odd prime number, has been awarded to Dr. P. FURTWÄNGLER, of Potsdam; and that offered by the philosophical faculty of the University of Göttingen for a study of surfaces possessing a family of closed geodesic lines has been awarded to Dr. OTTO ZOLL, of Düren.

AT its anniversary meeting, November 30, 1901, the Royal society of London made the first award of its Sylvester medal, conferring it upon Professor HENRI POINCARÉ for his many and important contributions to mathematical science.

OF the four Nobel prizes distributed at Stockholm in December, 1901, that for physics was awarded to Professor W. C. RÖNTGEN. Each prize was of the value of two hundred thousand francs.

AMONG the foreign members elected by the Göttingen academy of sciences in celebration of the one hundred and fifteenth anniversary of its foundation are Professor G. DARBOUX, of the University of Paris, as foreign member and Professors H. MINKOWSKI, of Zurich, and MICHEL LEVY, of Paris, as corresponding members.

DR. E. JAHNKE, of Berlin, has been made docent in the Technical school at Charlottenburg, presenting an inaugural lecture on "Solutions of rotational problems." He will offer a course of lectures on "Applications of Grassmann's methods."

DR. C. MOSER has been promoted to an assistant professorship of mathematics at the University of Berne.

DR. RAU, engineer with the firm of Schukert and Company, Nürnberg, has been appointed assistant professor of applied mathematics at the University of Jena.

MR. F. B. LITTELL has been appointed to a professorship of mathematics in the United States Navy.

MR. A. M. KENYON has been promoted to an associate professorship of mathematics at Purdue University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABHANDLUNGEN aus den Gebieten der Mathematik, Physik, Chemie und beschreibenden Naturwissenschaften. Festschrift zur Feier des 70sten Geburtstages von Richard Dedekind. Mit Beiträgen von H. Beckurts, R. Blasius, W. Blasius, G. Bodländer, G. Frerichs, R. Fricke, R. Meyer, R. Müller, H. Weber, A. Wernicke. Braunschweig, Vieweg, 1901. 8vo. 7+254 pp., 1 plate. M. 6.00

ARCAIS (F. D'). Corso di calcolo infinitesimale. Vol. II (ultimo). 2a edizione, con aggiunte e modificazioni. Padova, Draghi, 1901.

BECKURTS (H.). See ABHANDLUNGEN.

BENDT (F.). Katechismus der Differential- und Integralrechnung. 2te Auflage. Leipzig, Weber, 1901. 12mo. 16 + 268 pp. Cloth. (Weber's illustrierte Katechismen, No. 157.) M. 3.00

——. Katechismus der algebraischen Analysis. Leipzig, Weber, 1901. 12mo. 11 + 153 pp. Cloth. (Weber's illustrierte Katechismen, No. 229.) M. 2.50

BERNSTEIN (F.). Untersuchungen aus der Mengenlehre. (Diss., Halle.) Göttingen, Vandenhoeck & Ruprecht, 1901. 8vo. 54 pp. M. 1.20

BLASIUS (R.), BLASIUS (W.), BODLÄNDER (G.). See ABHANDLUNGEN.

BOLOMBURU (N. DE). Apuntes y ejercicios de geometría analítica, arreglados al nuevo programa de ingreso en la Escuela especial de ingenieros de minas. Madrid, Teódoro, 1901. 8vo. 120 pp., 3 plates. Fr. 6.00

BOTTARI (A.). Razionalità dei piani multipli $\{x, y, \sqrt{F(x, y)}\}$; nota. Bologna, Pongetti, 1901. 8vo. 7 pp.

BRENDEL (M.). Ueber partielle Integration. 1901. 8vo. (*Mathematische Annalen* 55, pp. 248–256.)

DICKSON (L. E.). Linear groups, with an exposition of the Galois field theory. Leipzig, Teubner, 1901. 8vo. 10 + 312 pp. Cloth. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. 6.) M. 12.00

DÖLP (H.). Aufgaben zur Differential- und Integralrechnung nebst den Resultaten und den zur Lösung nötigen theoretischen Erläuterungen; neu bearbeitet von E. Netto. 9te Auflage. Giessen, Ricker, 1901. 8vo. 3 + 216 pp. Cloth. M. 4.00

FRERICHS (G.), FRICKE (R.). See ABHANDLUNGEN.

GAUSS (C. F.). Sechs Beweise des Fundamentaltheorems über quadratische Reste; herausgegeben von E. Netto. Leipzig, Engelmann, 1901. 12mo. 111 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 122.) M. 1.80

HALSTED (G. B.). Supplementary report on noneuclidean geometry. 1901. (*Science* (n. s.) 14, pp. 705–717.)

HATHAWAY (A. S.). A primer of calculus. New York, Macmillan, 1901. 12mo. 8 + 139 pp. Cloth. $1.25

JAHRBUCH über die Fortschritte der Mathematik, begründet durch C. Ohrtmann; unter besonderer Mitwirkung von F. Müller und A. Wangerin herausgegeben von E. Lampe und G. Wallenberg. Vol. 30: Jahrgang 1899. (In 3 Heften.) Heft II. Berlin, Reimer, 1901. 8vo. Pp. 433–608. M.5.60

KLEIN (F.). Bericht über den Stand der Herausgabe von Gauss' Werken. Vierter Bericht. 1901. 8vo. (*Nachrichten der k. Gesellschaft der Wissenschaften zu Göttingen*, Geschäftliche Mittheilungen, 1901, pp. 1–4.)

LAMPE (E.). See JAHRBUCH.

MARASCO (G. B.). Sopra una particolare superficie del sesto ordine. Acireale, Tipografia dell' *Etna*, 1901. 8vo. 18 pp.

MARLETTA (G.). Sulle varietà del quarto ordine con un piano doppio nello spazio a quattro dimensioni. Catania, Giannotta, 1901. 8vo. 46 pp. Fr. 3.00

MARXSEN (S.). Ueber eine allgemeine Gattung irrationaler Invarianten und Covarianten für eine binäre Form ungerader Ordnung. (Diss.) Göttingen, 1900. 8vo. 50 pp.

MEYER (R.). See ABHANDLUNGEN.

MEYER (W. F.). Differenzial- und Integralrechnung. Vol I.: Differenzialrechnung. Leipzig, Göschen, 1901. 8vo. 18 + 395 pp. Cloth. (Sammlung Schubert, No. X.) M. 9.00

MÜLLER (F.). See JAHRBUCH.

MÜLLER (R.). See ABHANDLUNGEN.

NETTO (E.). See DÖLP (H.) and GAUSS (C. F.).

POGGI (F.). La serie di Lagrange; studio storico critico. Genova, Istituto Sordomuti, 1901. 8vo. 15 pp.

ROMEO (F.). Della congruenza generata dalle rette che si appoggiano ai raggi corrispondenti di tre fasci proiettivi contenuti in tre piani non appartenenti ad un fascio. Cosenza, 1901. 8vo. 22 pp.

ROST (G.). Theorie der Riemannschen Thetafunktion. Leipzig, Teubner, 1901. 4to. 4 + 66 pp. M. 4.00

RÜCKLE (G.). Quadratische Reciprocitätsgesetze in algebraischen Zahlkörpern. (Diss.) Göttingen, Vandenhoeck & Ruprecht, 1901. 8vo. 49 pp. M. 1.20

STAEBLE (F). Untersuchung der Flächen, deren Krümmungslinien bei orthogonaler Projektion auf eine andere Fläche wieder Krümmungslinien werden. (Diss.) München, 1901. 8vo. 30 pp.

STEINER (J.). Einige geometrische Betrachtungen (1826). Herausgegeben von R. Sturm. Leipzig, Engelmann, 1901. 12mo. 125 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 123.) M. 2.00

STURM (R.). See STEINER (J.).

VAN VLECK (E. B.). On the convergence of the continued fraction of Gauss and other continued fractions. 1901. 4to. (*Annals of Mathematics* (2) 3, pp. 1–18.)

——. On the convergence and character of the continued fraction $\frac{a_1 z}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots$. 1901. 4to. (*Transactions of the American Mathematical Society*, 2, pp. 476–483.)

WALLENBERG (G.), WANGERIN (A.). See JAHRBUCH.

WEBER (H.), WERNICKE (A.). See ABHANDLUNGEN.

WIELEITNER (H.). Ueber die Flächen dritter Ordnung mit Ovalpunkten. (Diss.) München, 1901. 8vo. 44 pp., 1 plate.

II. ELEMENTARY MATHEMATICS.

ALEU (M. L.). Pizarras de aritmética y algebra para uso de los aspirantes á ingreso en las academias de infantería, caballería, artillería, ingenieros, administración militar y armada, dispuestas con sujeción á las obras declaradas de texto. Madrid, Velasco, 1901. 8vo. 197 pp.

ALVAREZ DE CASTRO (A.). Soluzione grafica della trisezione dell'angolo. Roma, Giovanni, 1901. 8vo. 8 pp., 2 plates.

BARDEY (E.). Resultate und Auflösungen zu E. Bardey's Arithmetischen Aufgaben nebst Lehrbuch der Arithmetik vorzugsweise für Realschulen, Progymnasien und Realgymnasien. (In alter und neuer Ausgabe.) Neue Ausgabe, nach der 10ten Auflage bearbeitet von F. Pietzker und O. Presler. Leipzig, Teubner, 1901. 8vo. 4 + 128 pp. M. 1.00

BIDDLE (D.). Mathematical questions and answers from the *Educational Times*, Vol. 75. London, Hodgson, 1901. 12mo. Cloth. 6s. 6d.

BIELER. See SCHUSTER (M.).

BOURLET (C.). Cours de mathématiques, à l'usage des élèves architectes et ingénieurs, professé à l'Ecole des beaux-arts. Paris, Naud, 1901. 8vo. 3+249 pp.

BRADLY (S. R. N.). See LANGLEY (E. M.).

CATALÁN Y MONROY (F.). Tratado elemental de aritmética y geometría. Primera parte: aritmética. Logroño, Librería de *El Riojano*, 1901. 8vo. 166 pp.

F. (F.). Manuel d'algèbre et de trigonométrie. Paris, Poussielgue [1901]. 16mo. 8 + 216 pp.

FERRARIO (L.). Trattato di trigonometria piana. Milano, Sonzogno, 1901. 16mo. 62 pp. Fr. 0.15

FISHER (G. E.) and SCHWATT (I. J.). Complete secondary algebra. (New issue.) New York, Macmillan, 1901. 12mo. 12 + 564 + 18 pp. Half leather. $1.35

——. Elements of algebra, with exercises. (New issue.) New York, Macmillan, 1901. 12mo. 16 + 478 pp. Half leather. $1.10

——. Higher algebra. (New issue.) New York, Macmillan, 1901. 12mo. 18+615+18 pp. Half leather. $1.50

——. Rudiments of algebra. (New issue.) New York, Macmillan, 1901. 12mo. 5+183 pp. Half leather. $0.60

——. School algebra, with exercises. 2d edition. New York, Macmillan, 1901. 12mo. 14+407 pp. Half leather. $1.00

——. Secondary algebra. New York, Macmillan, 1901. 12mo. 10+442+18 pp. Half leather. $1.08

——. Text-book of algebra, with exercises; for secondary schools and colleges. Part I. New York, Macmillan, 1901. 12mo. 13+683 pp. Cloth. $1.40

French (C. H.) and Osborn (G.). Elementary algebra (to logarithms). 2d edition. With answers. London, Churchill, 1901. 12mo. 410 pp. Cloth. 4s. 6d.

Gies (W.). Was muss man von der Trigonometrie wissen? Allgemeinverständlich dargestellt. Berlin, Steinitz, 1901. 8vo. 75 pp. M. 1.00

Jacquemart (A.). Algèbre pratique en quinze leçons, à l'usage des écoles primaires supérieures et des écoles normales. Livre de l'élève. 4e edition. Paris, Larousse [1901]. 16mo. 112 pp. Fr. 1.00

Kutnewsky (M). See Müller (H.).

Langley (E. M.) and Bradly (S. R. N.). Algebra, adapted to the requirements of the first stage of the directory of the board of education. London, Murray, 1901. 12mo. 204 pp. Cloth. 1s. 6d.

Müller (H.) und Kutnewsky (M.). Sammlung von Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. Teil II. Ausgabe A: Für die oberen Klassen der Gymnasien. 8+347 pp. Cloth. M. 3.40

——. Teil II. Ausgabe B: Für reale Anstalten und Reformschulen. 8+360 pp. Cloth. M. 3.40

Müller (H.). Die Mathematik auf den Gymnasien und Realschulen; für den Unterricht dargestellt. Teil I: Die Unterstufe (Lehraufgabe der Klassen Quarta bis Unter-Sekunda). 2te Auflage. Ausgabe A: Für Gymnasien und Progymnasien. Leipzig, Teubner, 1902. 8vo. 7+137 pp. Cloth. M. 1.60

——. Ausgabe B: Für reale Anstalten und Reformschulen. Leipzig, Teubner, 1902. 8vo. 8+199 pp. Cloth. M. 2.20

Nouvelles Tables de logarithmes à cinq décimales pour les lignes trigonométriques dans les deux systèmes de la division centésimale et de la division sexagésimale du quadrant et pour les nombres de 1 à 12,000. Edition spéciale à l'usage des candidats aux Ecoles polytechnique et de Saint-Cyr. Paris, Gauthier-Villars, 1901. 8vo. 221 pp. Fr. 3.00

Niehus (P.). Auflösungen der Aufgaben in den Elementen der Arithmetik und der Algebra für Baugewerkschulen, Maschinenbauschulen und Handwerkerschulen; nebst Hinweisen zu den Lösungen. Magdeburg, Friese, 1901. 8vo. 31 pp. M. 1.00

OSBORN (G.). See FRENCH (C. H.).

PASQUALI (P.). Geometria intuitiva ad uso delle scuole elementari superiori, tecniche, normali e industriali: lezioni di ritaglio geometrico date al r. corso normale di Ripatransone. Parte I. Edizione migliorata e corretta. Parma, Battei, 1901. 16mo. 51 pp. Fr. 0.50

PIETZKER (F.) and PRESLER (O.). See BARDEY (E.).

RATTER (L.). Geometrische Aufgaben und Beispiele in rationalen Zahlen. (Progr.) Mähr.-Schönberg, 1901. 8vo.

RÜEFLI (J.). Lehrbuch der ebenen Trigonometrie. Bern, Schmid & Francke, 1902. 8vo. 31 pp. M. 0.80

SCHUSTER (A.). Mathematik für jedermann. Leichtfassliche Einführung in die niedere und höhere Mathematik. Stuttgart, Union, 1901. 8vo. 12 + 228 pp. M. 3.60

SCHUSTER (M.). Geometrische Aufgaben und Lehrbuch der Geometrie für Mittelschulen. Unter Mitwirkung von Dr. Bieler bearbeitet. Leipzig, Teuber, 1901. 8vo. 8 + 88 pp. Cloth. M. 1.40

SCHWATT (I. J.). See FISHER (G. E.).

TAYLOR (F. G.). An introduction to the practical use of logarithms, with examples in mensuration and answers to exercises. New York, Longmans, 1901. 8vo. 8 + 64 pp. $0.50

THOMSON (W.). Elementary algebra. London, Chambers, 1901. 12mo. 288 pp. Cloth. 2s.

TÖDTER (H.). Anfangsgründe der Arithmetik und Algebra. Für den Schul- und Selbstunterricht in entwickelnder Lehrform bearbeitet. Teil I. Ausgabe A. 4te Auflage. Bielefeld, Velhagen & Klasing, 1901. 8vo. 5 + 131 pp. M. 1.60

VERONESE (G.). Nozioni elementari di geometria intuitiva ad uso dei ginnasi inferiori. Verona, Drucker, 1901. 8vo. 8 + 80 pp. Fr. 1.00

WOLF (H.). Leitfaden für den Unterricht in Geometrie an Baugewerkschulen und anderen gewerblichen Schulen. Leipzig, Brandstetter, 1901. 8vo. 40 pp. M. 0.75

III. APPLIED MATHEMATICS.

ARNOLD (E.). und LA COUR (J. L.). Beitrag zur Vorausberechnung und Untersuchung von Ein- und Mehrphasenstromgeneratoren. Stuttgart, Enke, 1901. 8vo. 3 + 108 pp. (Sammlung elektrotechnischer Vorträge.) M. 3.60

BAHRDT (W.). Ueber die Bewegung eines Punktes auf einer rauhen Fläche, inbesondere auf einem rauhen Kreiscylinder und einem rauhen Kreiskegel. (Diss.) Kiel, 1901. 8vo. 47 pp.

BAULE (A.). Lehrbuch der Vermessungskunde. 2te Auflage. Leipzig, Teubner, 1901. 8vo. 8 + 471 pp. M. 8.80

BERNHARD (M.). Darstellende Geometrie mit Einschluss der Schattenkonstruktionen. Als Leitfaden für den Unterricht an technischen Lehranstalten, Oberrealschulen und Realgymnasien, sowie zum Selbstunterricht herausgegeben. Stuttgart, Enderlen, 1901. 8vo. 8 + 195 pp. M. 4.60

BIDLINGMAIER (F.). Geometrischer Beitrag zur Piëzoelektrizität der Krystalle. (Diss.) Göttingen, 1901. 8vo. 60 pp., 1 plate.

BOVEY (H. F.). A treatise on hydraulics. 2d edition, rewritten. New York, Wiley, 1901. 8vo. 18 + 582 pp. Cloth. $5.00

CHOURA (J.). Lehrbuch der darstellenden Geometrie für die k. und k. Kadettenschule und die k. und k. Militär-Oberrealschule. Wien, Seidel, 1901. 8vo. 3 + 303 pp. Cloth. M. 4.00

CLASSEN (J.). Mathematische Optik. Leipzig, Göschen, 1901. 8vo. 10 + 207 pp. Cloth. (Sammlung Schubert, No. XI.) M. 6.00

COFFIN (F. C.). The graphical solution of hydraulic problems; treating of the flow of water through pipes, in channels and sewers, over weirs, etc. 2d edition. New York, Wiley, 1901. 12mo. 5 + 79 pp. Morocco. $2.50

DOBBS (W. J.). A treatise on elementary statics for the use of schools and colleges. New York, Macmillan, 1901. 12mo. 11 + 311 pp. Cloth. $2.00

DOLBERG (F.). Theorie der Wärmeleitung in einem homogenen, schalenförmigen Körper, der von zwei nicht konzentrischen Kugelflächen begrenzt wird. (Diss.) München, 1900. 8vo. 39 pp.

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